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ABSTRACT. The occurrence of a finite time singularity is shown for a free boundary problem modeling microelectromechanical systems (MEMS) when the applied voltage exceeds some value. The model involves a singular nonlocal reaction term and a nonlinear curvature term accounting for large deformations.

1. INTRODUCTION

An idealized electostatically actuated microelectromechanical system (MEMS) consists of a fixed horizontal ground plate held at zero potential above which an elastic membrane held at potential V is suspended. A Coulomb force is generated by the potential difference across the device and results in a deformation of the membrane, thereby converting electrostatic energy into mechanical energy, see [1, 4, 7] for a more detailed account and further references. After a suitable scaling and assuming homogeneity in transversal horizontal direction, the ground plate is assumed to be located at $z = -1$ and the membrane displacement $u = u(t,x) \in (-1,\infty)$ with $t > 0$ and $x \in I := (-1,1)$ evolves according to

$$\partial_t u - \partial_x \left(\frac{\partial_x u}{\sqrt{1+\varepsilon^2(\partial_x u)^2}} \right) = -\lambda \left(\varepsilon^2 \, |\partial_x \psi(t,x,u(t,x))|^2 + |\partial_z \psi(t,x,u(t,x))|^2 \right) , \tag{1}$$

for $t > 0$ and $x \in I$ with boundary conditions

$$u(t,\pm 1) = 0 , \quad t > 0 , \tag{2}$$

and initial condition

$$u(0,x) = u^0(x) , \quad x \in I . \tag{3}$$

The electrostatic potential $\psi = \psi(t,x,z)$ satisfies a rescaled Laplace equation in the region

$$\Omega(u(t)) := \{(x,z) \in I \times (-1,\infty) \ : \ -1 < z < u(t,x)\}$$

between the plate and the membrane which reads

$$\varepsilon^2 \, \partial_x^2 \psi + \partial_z^2 \psi = 0 , \quad (x,z) \in \Omega(u(t)) , \quad t > 0 , \tag{4}$$

$$\psi(t,x,z) = \frac{1+z}{1+u(t,x)} , \quad (x,z) \in \partial\Omega(u(t)) , \quad t > 0 , \tag{5}$$

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where $\varepsilon > 0$ denotes the aspect ratio of the device and $\lambda > 0$ is proportional to the square of the applied voltage. The dynamics of (u, ψ) is thus given by the coupling of a quasilinear parabolic equation for u and an elliptic equation in a moving domain for ψ, the latter being only well-defined as long as the membrane does not touch down on the ground plate, that is, u does not reach the value -1. To guarantee optimal operating conditions of the device, this touchdown phenomenon has to be controlled and its occurrence is obviously related to the value of λ.

The main difficulty to be overcome in the analysis of (1)-(5) is the nonlocal and nonlinear implicit dependence on u of the right-hand side of (1) which is also singular if u approaches -1. Except for the singularity, these features disappear when setting $\varepsilon = 0$ in (1)-(5), a commonly made assumption which reduces (1)-(5) to a singular semilinear reaction-diffusion equation. This so-called small aspect ratio model has received considerable attention in recent years, see [4, 7] and the references therein. In this simplified situation, it has been established that touchdown does not take place if λ is below a certain threshold value $\lambda_* > 0$, but occurs if λ exceeds this value [4, 5, 6].

We have recently investigated the well-posedness of (1)-(5) and established the following result [3].

Theorem 1 (**Local Well-Posedness**). *Let $q \in (2, \infty)$, $\varepsilon > 0$, $\lambda > 0$, and consider an initial value*

$$u^0 \in W_q^2(I) \quad \text{such that} \quad u^0(\pm 1) = 0 \quad \text{and} \quad 0 \geq u^0(x) > -1 \quad \text{for} \quad x \in I\ . \tag{6}$$

Then there is a unique maximal solution (u, ψ) to (1)-(5) on the maximal interval of existence $[0, T_m^\varepsilon)$ in the sense that

$$u \in C^1\big([0, T_m^\varepsilon), L_q(I)\big) \cap C\big([0, T_m^\varepsilon), W_q^2(I)\big)$$

satisfies (1)-(3) together with

$$0 \geq u(t, x) > -1\ , \quad (t, x) \in [0, T_m^\varepsilon) \times I\ , \tag{7}$$

and $\psi(t) \in W_2^2\big(\Omega(u(t))\big)$ solves (4)-(5) for each $t \in [0, T_m^\varepsilon)$.

We have also shown in [3] that, if λ and u^0 are sufficiently small, the solution (u, ψ) to (1)-(5) exists for all times (i.e. $T_m^\varepsilon = \infty$) and touchdown does not take place, not even in infinite time.

Theorem 2 (**Global Existence**). *Let $q \in (2, \infty)$, $\varepsilon > 0$, and consider an initial value u^0 satisfying (6). Given $\kappa \in (0, 1)$, there are $\lambda_*(\kappa) > 0$ and $r(\kappa) > 0$ such that, if $\lambda \in (0, \lambda_*(\kappa))$ and $\|u^0\|_{W_q^2(I)} \leq r(\kappa)$, the maximal solution (u, ψ) to (1)-(5) exists for all times and $u(t, x) \geq -1 + \kappa$ for $(t, x) \in [0, \infty) \times I$.*

On the other hand, we have been able to prove that no stationary solution to (1)-(5) exists provided λ is sufficiently large. However,whether or not T_m^ε is finite in this case has been left as an open question. The purpose of this note is to show that – as expected on physical grounds – T_m^ε is indeed finite for λ sufficiently large.

Theorem 3 (**Finite time singularity**). *Let $q \in (2, \infty)$, $\varepsilon > 0$, and consider an initial value u^0 satisfying (6). If $\lambda > 1/\varepsilon$ and (u, ψ) denotes the maximal solution to (1)-(5) defined on $[0, T_m^\varepsilon)$, then $T_m^\varepsilon < \infty$.*

The criterion $\lambda > 1/\varepsilon$ is likely to be far from optimal. As we shall see below, improving it would require to have a better control on $\partial_x u(\pm 1)$. The proof of Theorem 3 relies on the derivation of a chain of estimates which allow us to obtain a lower bound on the L_1-norm of the right-hand side of (1) depending only on u. The lower bound thus obtained is in fact the mean value of a convex function of u, and we may then end the proof with the help of Jensen's inequality, an argument which has already been used for the small aspect ratio model, see [5, 6].

We shall point out that, in contrast to the small aspect ratio model, the finiteness of T_m^ε does not guarantee that the touchdown phenomenon really takes place as $t \to T_m^\varepsilon$. Indeed, according to [3, Theorem 1.1 (ii)], the finiteness of T_m^ε implies that $\min_{[-1,1]} u(t) \longrightarrow -1$ or $\|u(t)\|_{W_q^2(I)} \longrightarrow \infty$ as $t \to T_m^\varepsilon$. While the former corresponds to the touchdown behaviour, the latter is more likely to be interpreted as the membrane being no longer the graph of a function at time T_m^ε.

2. Proof of Theorem 3

Let $q \in (2,\infty)$, $\varepsilon > 0$, $\lambda > 0$ and consider an initial value u^0 satisfying (6). We denote the maximal solution to (1)-(5) defined on $[0, T_m^\varepsilon)$ by (u, ψ). Differentiating the boundary conditions (5), we readily obtain

$$\partial_x \psi(t,x,-1) = \partial_x \psi(t,x,u(t,x)) + \partial_x u(t,x)\, \partial_z \psi(t,x,u(t,x)) = 0 \ , \qquad (t,x) \in (0,T_m^\varepsilon) \times I \ , \tag{8}$$

and

$$\partial_z \psi(t,\pm 1,z) = 1 \ , \qquad (t,z) \in (0,T_m^\varepsilon) \times (-1,0) \ . \tag{9}$$

Additional information on the boundary behaviour of ψ is provided by the next lemma.

Lemma 4. *For $t \in (0,T_m^\varepsilon)$,*

$$1+z \le \psi(t,x,z) \le 1 \ , \qquad (x,z) \in \Omega(u(t)) \ , \tag{10}$$

$$\pm \partial_x \psi(t,\pm 1,z) \le 0 \ , \qquad z \in (-1,0) \ . \tag{11}$$

Proof. Fix $t \in (0,T_m^\varepsilon)$. The upper bound in (10) readily follows from the maximum principle. Next, the function σ, defined by $\sigma(x,z) = 1+z$, obviously satisfies $\varepsilon^2 \partial_x^2 \sigma + \partial_z^2 \sigma = 0$ in $\Omega(u(t))$ as well as

$$\begin{aligned} &\sigma(\pm 1,z) = 1+z = \psi(t,\pm 1,z) \ , && z \in (-1,0) \ , \\ &\sigma(x,-1) = 0 = \psi(t,x,-1) \ , && x \in (-1,1) \ . \end{aligned}$$

Owing to the non-positivity (7) of $u(t)$, it also satisfies

$$\sigma(x,u(t,x)) = 1 + u(t,x) \le 1 = \psi(t,x,u(t,x)) \ , \qquad x \in (-1,1) \ ,$$

and we infer from the comparison principle that $\psi(t,x,z) \ge \sigma(x,z)$ for $(x,z) \in \Omega(u(t))$. It then follows from (10) that $\psi(t,x,z) \ge 1+z = \psi(t,\pm 1,z)$ for $(x,z) \in \Omega(u(t))$ which readily implies (11). □

To simplify notations, we set

$$\gamma_m(t,x) := \partial_z \psi(t,x,u(t,x)) \ , \quad \gamma_g(t,x) := \partial_z \psi(t,x,-1) \ , \qquad (t,x) \in (0,T_m^\varepsilon) \times (-1,1) \ , \tag{12}$$

and first derive an upper bound of the L_1-norm of the right-hand side of (1), observing that, due to (8), it also reads

$$-\lambda\varepsilon^2\ |\partial_x\psi(t,x,u(t,x))|^2 + |\partial_z\psi(t,x,u(t,x))|^2 = -\lambda\left(1+\varepsilon^2(\partial_x u(t,x))^2\right)\gamma_m(t,x)^2\ .$$

Lemma 5. *For $t\in(0,T_m^\varepsilon)$,*

$$\int_{-1}^1 \left(1+\varepsilon^2(\partial_x u(t,x))^2\right)\gamma_m(t,x)^2\ \mathrm{d}x \ge 2\int_{-1}^1 \left(1+\varepsilon^2(\partial_x u(t,x))^2\right)\gamma_m(t,x)\ \mathrm{d}x - 2\ . \tag{13}$$

Proof. Fix $t\in(0,T_m^\varepsilon)$. We multiply (4) by $\partial_z\psi(t)-1$ and integrate over $\Omega(u(t))$. Using (8), (9), and Green's formula we obtain

$$\begin{aligned} 0 = &-\varepsilon^2\int_{\Omega(u)} \partial_x\partial_z\psi\ \partial_x\psi\ \mathrm{d}(x,z) + \varepsilon^2\int_{-1}^1 (\partial_x u)^2\ \gamma_m\,(\gamma_m-1)\ \mathrm{d}x \\ &- \frac{1}{2}\int_{-1}^1 \left(\gamma_g^2-2\gamma_g\right)\ \mathrm{d}x + \frac{1}{2}\int_{-1}^1 \left(\gamma_m^2-2\gamma_m\right)\ \mathrm{d}x\ . \end{aligned}$$

Since

$$\int_{\Omega(u)} \partial_x\partial_z\psi\ \partial_x\psi\ \mathrm{d}(x,z) = \frac{1}{2}\int_{-1}^1 (\partial_x u)^2\ \gamma_m^2\ \mathrm{d}x$$

by (8) and since $\gamma_g^2-2\gamma_g\ge -1$, we end up with (13). □

We again use (4) to obtain a lower bound for the boundary integral of the right-hand side of (13) which depends on the Dirichlet energy of ψ.

Lemma 6. *For $t\in(0,T_m^\varepsilon)$,*

$$\int_{-1}^1 \left(1+\varepsilon^2(\partial_x u(t,x))^2\right)\gamma_m(t,x)\ \mathrm{d}x \ge \int_{\Omega(u(t))} \left(\varepsilon^2|\partial_x\psi(t,x,z)|^2 + |\partial_z\psi(t,x,z)|^2\right)\ \mathrm{d}(x,z)\ . \tag{14}$$

Proof. Fix $t\in(0,T_m^\varepsilon)$. We multiply (4) by $\psi(t)$ and integrate over $\Omega(u(t))$. Using (5), (8), and Green's formula we obtain

$$\begin{aligned} 0 = &-\int_{\Omega(u(t))} \left(\varepsilon^2|\partial_x\psi(t,x,z)|^2 + |\partial_z\psi(t,x,z)|^2\right)\ \mathrm{d}(x,z) + \varepsilon^2\int_{-1}^0 (1+z)\ \partial_x\psi(t,1,z)\ \mathrm{d}z \\ &-\varepsilon^2\int_{-1}^0 (1+z)\ \partial_x\psi(t,-1,z)\ \mathrm{d}z + \varepsilon^2\int_{-1}^1 (\partial_x u(t,x))^2\ \gamma_m(t,x)\ \mathrm{d}x + \int_{-1}^1 \gamma_m(t,x)\ \mathrm{d}x\ . \end{aligned}$$

Owing to (11), the second and third terms of the right-hand side of the above equality are non-positive, whence (14). □

We finally argue as in [2, Lemma 9] to establish a connection between the Dirichlet energy of ψ and u.

Lemma 7. *For $t \in (0, T_m^\varepsilon)$,*

$$\int_{\Omega(u(t))} \left(\varepsilon^2 |\partial_x \psi(t,x,z)|^2 + |\partial_z \psi(t,x,z)|^2\right)\ \mathrm{d}(x,z) \ge \int_{-1}^1 \frac{\mathrm{d}x}{1+u(t,x)}\ . \tag{15}$$

Proof. Let $t \in (0, T_m^\varepsilon)$ and $x \in (-1,1)$. We deduce from (5) and the Cauchy-Schwarz inequality that

$$\begin{aligned}\frac{1}{1+u(t,x)} =& \frac{(\psi(t,x,u(t,x)) - \psi(t,x,-1))^2}{1+u(t,x)} = \frac{1}{1+u(t,x)} \left(\int_{-1}^{u(t,x)} \partial_z \psi(t,x,z)\ \mathrm{d}z \right)^2 \\ \le& \int_{-1}^{u(t,x)} \left(\partial_z \psi(t,x,z)\right)^2\ \mathrm{d}z\ . \end{aligned}\tag{16}$$

Integrating the above inequality with respect to $x \in (-1,1)$ readily gives (15). ☐

Remark 8. *Observe that* (16) *provides a quantitative estimate on the singularity of $\partial_z \psi$ generated by u when touchdown occurs.*

Combining the three lemmas above with Jensen's inequality give the following estimate.

Proposition 9. *For $t \in (0, T_m^\varepsilon)$,*

$$\int_{-1}^1 \left(1 + \varepsilon^2 (\partial_x u(t,x))^2\right) \gamma_m(t,x)^2\ \mathrm{d}x \ge 4\varphi\left(\frac{1}{2} \int_{-1}^1 u(t,x)\ \mathrm{d}x \right) - 2\ , \tag{17}$$

where $\varphi(r) := 1/(1+r)$, $r \in (-1,\infty)$.

Proof. Fix $t \in (0, T_m^\varepsilon)$. We infer from Lemma 5, Lemma 6, and Lemma 7 that

$$\int_{-1}^1 \left(1 + \varepsilon^2 (\partial_x u(t,x))^2\right) \gamma_m(t,x)^2\ \mathrm{d}x \ge 2 \int_{-1}^1 \varphi(u(t,x))\ \mathrm{d}x - 2\ .$$

To complete the proof, we argue as in [5, 6] and use the convexity of φ and Jensen's inequality to obtain (17). ☐

Proof of Theorem 3. Introducing

$$E(t) := -\frac{1}{2} \int_{-1}^1 u(t,x)\ \mathrm{d}x\ , \qquad t \in [0, T_m^\varepsilon)\ ,$$

the bounds (7) ensure that

$$0 \le E(t) < 1\ , \qquad t \in [0, T_m^\varepsilon)\ . \tag{18}$$

It follows from (1), (8), and Proposition 9 that

$$\begin{aligned}\frac{dE}{dt}(t) &= -\frac{1}{2}\left[\frac{\partial_x u(t,x)}{\sqrt{1+\varepsilon^2\left(\partial_x u(t,x)\right)^2}}\right]_{x=-1}^{x=1} + \frac{\lambda}{2}\int_{-1}^{1}\left(1+\varepsilon^2(\partial_x u(t,x))^2\right)\gamma_m(t,x)^2\,\mathrm{d}x \\ &\geq F_\lambda(E) := 2\lambda\varphi(-E) - \lambda - \frac{1}{\varepsilon}\ . \end{aligned} \tag{19}$$

If $\lambda > 1/\varepsilon$, we note that $F_\lambda(0) > 0$ and thus $F_\lambda(r) \geq F_\lambda(0) > 0$ for $r \in [0,1)$ due to the monotonicity of F_λ. Since $E(0) \geq 0$ by (18), it follows from (19) and the properties of F_λ that $t \mapsto E(t)$ is increasing on $[0, T_m^\varepsilon)$. Consequently,

$$\frac{dE}{dt}(t) \geq F_\lambda(E(0)) \geq F_\lambda(0)\ , \qquad t \in [0, T_m^\varepsilon)\ .$$

Integrating the previous inequality with respect to time and using (18), we end up with the inequality $1 \geq E(0) + F_\lambda(0) T_m^\varepsilon$ which provides the claimed finiteness of T_m^ε. □

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References

[1] N.D. Brubaker and J.A. Pelesko. Non-linear effects on canonical MEMS models. *European J. Appl. Math* **22** (2011), 455–470.
[2] J. Escher, Ph. Laurençot, and Ch. Walker. A parabolic free boundary problem modeling electrostatic MEMS. *Arch. Rational Mech. Anal.* (to appear).
[3] J. Escher, Ph. Laurençot, and Ch. Walker. Dynamics of a free boundary problem with curvature modeling electrostatic MEMS. `arXiv:1302.6026` (submitted).
[4] P. Esposito, N. Ghoussoub, and Y. Guo. *Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS*, volume **20** of *Courant Lecture Notes in Mathematics*. Courant Institute of Mathematical Sciences, New York, 2010.
[5] G. Flores, G. Mercado, J. A. Pelesko, and N. Smyth. Analysis of the dynamics and touchdown in a model of electrostatic MEMS. *SIAM J. Appl. Math.* **67** (2007), 434–446.
[6] Y. Guo, Z. Pan, and M.J. Ward. Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties. *SIAM J. Appl. Math.* **66** (2005), 309–338.
[7] J.A. Pelesko and D.H. Bernstein. *Modeling MEMS and NEMS*. Chapman & Hall/CRC, Boca Raton, FL, 2003.

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